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Analyticity of the Density of States and Replica Method for Random Schrödinger Operators on a Lattice

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We analyze the density of states and some aspects of the replica method for Anderson's tight binding model on a lattice of arbitrary dimension, with diagonal disorder. We give heuristic arguments for the conjectures that the classical value of the exponent ν of the localization length is $1/2$ and that the upper critical dimension, d_c^{loc}, is bounded by $4 \leqq d_c^{\mathrm{loc}} \leqq 6$.

KEY WORDS: Tight binding model; density of states; replica method.

1. INTRODUCTION

The Anderson tight binding model[1] describes a quantum mechanical particle moving in a random potential on the lattice $\mathbb{Z}^\nu$. The dynamics of such a particle is described by the Hamiltonian

$$H = -\Delta + V \text{ acting on the Hilbert space } l_2(\mathbb{Z}^\nu) \tag{1.1}$$

where Δ is the finite difference Laplacian, and V is a real random potential belonging to the probability space

$$\mathscr{R} = \mathop{\times}_{j \in \mathbb{Z}^\nu} (\mathbb{R}, d\lambda(V(j))) \tag{1.2}$$

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where $d\lambda$ is a probability measure prescribing the distribution of values $V(j)$ of the potential V at j. Throughout this paper it is assumed that $d\lambda$ is independent of j and that it is absolutely continuous with respect to Lebesgue measure, dV, with a smooth density.

Since every $V \in \mathscr{R}$ is self-adjoint on a dense domain $D(V)$ in $l_2(\mathbb{Z}^\nu)$ and since Δ is bounded, $H \equiv H_V$ is self-adjoint on $D(V)$.

We shall decompose the Laplacian according to

$$\begin{aligned} \Delta &= P - 2\nu\mathbb{1}, \qquad \text{where} \\ (Pf)(j) &= \sum_{|i-j|=1} f(i), \qquad i, j \in \mathbb{Z}^\nu \end{aligned} \tag{1.3}$$

Anderson's model is studied in connection with the theory of conductivity in media with random impurities. Roughly speaking, conducting states are associated with absolutely continuous spectrum and extended (plane-wave type) eigenfunctions of H_V, while insulating, or nonconducting, states are supposed to be associated with point spectrum of H_V—typically with exponentially decaying eigenfunctions.

It is well known that spectral properties of H_V are true with probability zero or one, with respect to $d\bar{\lambda}(V) \equiv \prod_j d\lambda(V(j))$,[2,3] The reason for this fact is that the group of translations $\{T_j : j \in \mathbb{Z}^\nu\}$, $(T_j f)(i) \equiv f(i-j)$, acts ergodically on $\mathscr{R}$ and that H_V and $H_{T_j V}$ are unitarily equivalent. Using these ideas it is easy to show, for example, that

$$\operatorname{spec} H_V = \operatorname{spec}(-\Delta) + \operatorname{supp} \lambda$$

for almost all $V \in \mathscr{R}$.[2]

Thus the primary questions about H_V concern the density of states of H_V, the nature of its spectrum and transport properties. In one dimension, $\operatorname{spec} H_V$ is known to be pure point, with exponentially localized eigenstates.[2,3] In higher dimensions ($\nu \geqslant 2$) the same type of spectrum is expected in the presence of high disorder, e.g.,

$$d\lambda(V) = \left(\frac{\gamma}{2\pi}\right)^{1/2} \exp\left[-\frac{\gamma}{2} V^2\right] dV \tag{1.4}$$

with $0 < \gamma \ll 1$, and near the edges of the spectrum. For mathematical results in this direction see Ref. 4.

For weak disorder, $\gamma \gg 1$, in three dimensions, it has been conjectured by Anderson[1] that there is an interval of (absolutely) continuous spectrum away from the edges of $\operatorname{spec} H_V$, outside of which pure point spectrum appears. The energies, E_c and $\bar{E}_c$, at which the type of the spectrum changes, are called mobility edges.

In this paper we study the density of states, $\rho(E)$, of H_V as a function of the energy $E \in \operatorname{spec} H_V$. We show that, for some class of distributions $d\lambda$ which are analytic in a strip of width $> 4\nu$ around the real axis, $\rho(E)$ is analytic in E, for $|\operatorname{Re} E| \gg 1$ and $|\operatorname{Im} E|$ small. Moreover, the decay of $\rho(E)$, as $E \to +\infty$, is governed by the one of $d\lambda$. In particular, for the Gaussian distribution defined in (1.4), $\rho(E)$ is analytic for $|E| \gg 1$, with Gaussian decay, as $|E| \to +\infty$. Moreover, for large disorder $0 < \gamma \ll 1$, we show that $\rho(E)$ is analytic in E in a domain containing the real axis. Related results have been established by Edwards and Thouless.[10] By a very simple, elegant analysis, Wegner has recently proven that for distributions $d\lambda$ of the form

$$d\lambda(V) = w(V)\,dV$$

where w is positive and bounded on an interval $I = \operatorname{supp} w$, $\rho(E)$ is positive and bounded on $\operatorname{spec} H_V = \operatorname{spec}(-\Delta) + I$.[14] Thus, $\rho(E)$ neither diverges nor vanishes at the mobility edges. Our results are compatible with the conjecture that $\rho(E)$ is real analytic in E when w is analytic in a strip around the real axis.

We now give a precise definition of the density of states: Let Λ be some finite sublattice of $\mathbb{Z}^\nu$, and $H_V(\Lambda)$ the Hamiltonian defined in (1.1), but with Δ replaced by Δ^Λ, where Δ^Λ is the finite difference Laplacian on $l_2(\Lambda)$ with suitable boundary conditions (Dirichlet or periodic) at $\partial\Lambda$, the boundary of Λ. Let $N_\Lambda(E; V)$ denote the number of eigenvalues of $H_V(\Lambda)$ less than or equal to E. The properties of $N_\Lambda(E; V)$ are discussed in Ref. 3. For our purposes it suffices to recall the following elementary result.

Proposition 1. With probability 1 with respect to $d\bar{\lambda}$

$$\lim_{\Lambda \nearrow \mathbb{Z}^\nu} \frac{1}{|\Lambda|} N_\Lambda(E; V) \equiv n(E)$$

exists, is increasing in E and independent of $V \in \mathscr{R}$, almost surely. Furthermore, by the results in Ref. 14,

$$n(E) = \int \langle x | P(E; V) | x \rangle \, d\bar{\lambda}(V)$$

where x is an arbitrary site of $\mathbb{Z}^\nu$, and $P(E; V)$ is the spectral projection onto the subspace of states of energy less than or equal to E. The density of states is defined as the measure

$$\rho(E) = \frac{d}{dE} n(E) = \frac{d}{dE} \int \langle x | P(E; V) | x \rangle \, d\bar{\lambda}(V) \tag{1.5}$$

Proofs of Proposition 1 may be found, for example, in Refs. 5 and 6.

The resolvent of H_V is defined, as usual, by

$$R_V(E+i\eta)\equiv[H_V-E-i\eta]^{-1}$$

and, formally,

$$dP(E;V)=\lim_{\eta\downarrow 0}\{R_V(E-i\eta)-R_V(E+i\eta)\}\,dE \tag{1.6}$$

In Section 2, we use the Neumann series expansion of R_V in the off-diagonal part, P, of H_V [see (1.3)] which converges when η is sufficiently large, in order to study the density of states, $\rho(E)$, at large values of $|E|$ or high disorder. See also Ref. 10. Our expansion is related to standard high-temperature expansions in statistical mechanics; (the role of temperature is played, here, by a combination of γ^{-1} and $|E|$). Many versions of that expansion have been used before; see, e.g., Refs. 7–9. After taking the average of R_V in $d\bar{\lambda}$; where $d\lambda(V)$ is analytic in a strip of width $>4\nu$ around the real axis in which it decays to 0—e.g., for $d\lambda$ given by (1.4)—our expansion converges even as $\eta\downarrow 0$, provided either $|E|\gg 1$ or there is large disorder, $0<\gamma\ll 1$. This permits us to justify (1.6) for matrix elements integrated with $d\bar{\lambda}$ and hence to establish decay—and analyticity properties of $\rho(E)$ in E.

The second topic of this paper is the *replica method.* For Gaussian $d\lambda$, this method relates the expectation value of R_V in $d\bar{\lambda}$ to the $N\to 0$ limit of the $O(N)$-invariant $g|\boldsymbol{\phi}|^4$ lattice field theory, with

$$\boldsymbol{\phi}=(\phi^1,\ldots,\phi^N),\qquad g\propto -\gamma^{-1} \tag{1.7}$$

and complex squared mass $z\equiv E+i\eta$, $\eta>0$ (see Section 3 for precise definitions):

$$\begin{aligned}
&\int d\bar{\lambda}(V)\langle x|R_V(z)|y\rangle\\
&\quad=\lim_{N\to 0} iZ_N^{-1}\int d\bar{\lambda}(V)\int d\boldsymbol{\phi}\phi^1(x)\phi^1(y)\\
&\qquad\times\exp\left[-\frac{i}{2}(\boldsymbol{\phi},(-\Delta+V-z)\boldsymbol{\phi})\right]\\
&\quad=\lim_{N\to 0} Z_N^{-1}\int d\boldsymbol{\phi}\phi^1(x)\phi^1(y)\\
&\qquad\times\exp\left[-\frac{1}{2}(\boldsymbol{\phi},(-\Delta+z)\boldsymbol{\phi})\right]\exp\left[(8\gamma)^{-1}\sum_{j\in\mathbb{Z}^\nu}|\boldsymbol{\phi}(j)|^4\right]
\end{aligned} \tag{1.8}$$

where

$$(\boldsymbol{\phi},A\boldsymbol{\phi})=\sum_{\alpha=1}^{N}\sum_{i,j\in\mathbb{Z}^\nu}\phi^\alpha(i)A(i,j)\phi^\alpha(j)$$

$$d\boldsymbol{\phi}=\prod_{\alpha=1}^{N}\prod_{j\in\mathbb{Z}^\nu}d\phi^\alpha(j)$$

and

$$Z_N = \int d\tilde{\lambda}(V) \int d\phi \exp\left[-\frac{i}{2}(\phi, (-\Delta + V - z)\phi) \right] \tag{1.9}$$

Mathematically, the right sides of (1.8) and (1.9) are first defined in a finite region $\Lambda \subset \mathbb{Z}^\nu$, and the left-hand side of (1.8) is then obtained by taking the limit $\Lambda \nearrow \mathbb{Z}^\nu$.

The technique of calculating quantities, like $\int d\tilde{\lambda}(V)\langle x \mid R_V(z) | y\rangle$, as limits of Euclidean Green's functions of N-component scalar field theories, as $N \to 0$, is known as the *replica method*; ($N = \#$ of replicas). It is frequently used in the study of disordered systems; see, e.g., Refs. 1 and 11. However, its range of validity appears to be limited, so that a mathematically rigorous study of that method seems desirable. We prove that the replica method is applicable in a calculation of the density of states—among other quantities—of random Schrödinger operators on a lattice, for large $|E|$ or large disorder, i.e., in the parameter region where localized states are expected.

The technique we shall use to analyze the replica method is a combination of random walk expansions[12] of $R_N(z)$ and $\det(-\Delta + V - z)^{-N/2}$, obtained from Neumann expansions in P, with a standard high-temperature expansion for systems with long-range interactions.[7,9] A similar analysis in the context of polymer physics is given in Ref. 13. In the course of our analysis we explicitly relate the expectation value of the resolvent R_V in $d\tilde{\lambda}$ to the $N \to 0$ limit of a two-point function of the $O(N)$–invariant $g|\phi|^4$ model with negative coupling constant and complex mass, as indicated in (1.8).

In Section 4 we propose and discuss some conjectures concerning the critical dimension of the Anderson model and the exponent ν of the localization length.

The material presented in Sections 2 and 3 of this paper was worked out between summer 1980 and winter 1981, but, for various reasons, was not written up as a paper. Large portions of it are of an expository nature! S. A. Molchanov has informed us that he has independently found related results.

2. THE RANDOM WALK EXPANSION

We start by deriving a random walk expansion for the matrix elements of the resolvent

$$\langle x | R_V(z) | y \rangle, \qquad x, y \in \mathbb{Z}^\nu, \quad \operatorname{Im} z \neq 0$$

Using (1.1) and (1.3) we may write H_V as

$$\begin{aligned} H_V - z &= D(z) - P, \qquad \text{where} \\ D(z) &= (2\nu - z)\mathbb{1} + V \end{aligned} \tag{2.1}$$

We expand in a Neumann series in P: Hence

$$R_V(z) = (D(z) - P)^{-1} = \sum_{n=0}^{\infty} D(z)^{-1}\big(PD(z)^{-1}\big)^n \tag{2.2}$$

Writing out each term on the right-hand side of (2.2) as a sum of products of matrix elements we obtain

$$\langle x|R_V(z)|y\rangle = \sum_{\omega\,:\,x\to y} \prod_{j\in\mathbb{Z}^\nu} D_j(z)^{-n_j(\omega)} \tag{2.3}$$

where ω is a random walk of nearest-neighbor steps starting at x and ending at y, $n_j(\omega)$ is the total number of visits of ω at the site $j \in \mathbb{Z}^\nu$, and

$$D_j(z) = 2\nu - z + V(j)$$

The expansions (2.2), (2.3) converge absolutely if

$$\|P[(2\nu - z)\mathbb{1} + V]^{-1}\| < 1$$

It is immediate that $\|P\| = 2\nu$, and

$$\|[(2\nu - z)\mathbb{1} + V]^{-1}\| \leqslant |\mathrm{Im}\, z|^{-1}$$

Thus, absolute convergence is assured if

$$|\mathrm{Im}\, z| > 2\nu \tag{2.4}$$

Next, we integrate both sides of (2.3), term by term, with $d\bar{\lambda}$. This yields

$$\int \langle x|R_V(z)|y\rangle\, d\bar{\lambda}(V)$$
$$= \sum_{\omega\,:\,x\to y} \prod_{j\in\mathbb{Z}^\nu} \int d\lambda(V)\left(\frac{1}{2\nu - z + V}\right)^{n_j(\omega)} \tag{2.5}$$

We propose to analytically continue each term on the right-hand side of (2.5) in z beyond the domain specified in (2.4) and prove absolute convergence of the analytically continued expansion. This yields an analytic continuation of the left-hand side of (2.5) in z. Since the density of states is the discontinuity of $\int\langle x|R_V(z)|x\rangle\, d\bar{\lambda}(V)$ along the real axis [see Proposition 1, (1.5), and (1.6)] the continued expansion will permit us to analyze $\rho(E)$, as well. In order to implement this program we make the following assumptions on $d\lambda$ (we do not aim at maximal generality):

(1) $$d\lambda(V) = w(V)\, dV \tag{2.6}$$

where

(2) w is analytic in V in the strip $\{V : |\mathrm{Im}\, V| < 2(\nu + \epsilon)\}$ (2.7)

for some arbitrarily small, but positive ϵ, and

(3)
$$p(E) \equiv \sup_{|V-E| \leqslant 2\nu+\epsilon} |w(V)| \to 0 \tag{2.8}$$

as $E \to \pm\infty$ (E real).

We now summarize our main results.

Theorem 2.1. Let $d\lambda$ satisfy (2.6)–(2.8), and let $z = E + i\eta$. Then (1) for $|\eta| \neq 0$, the expansion (2.5) converges absolutely, provided $|E|$ is large enough; (2) the density of states, $\rho(E)$, is real analytic in E, for $|\mathrm{Re}\, E|$ large enough, and

$$0 < \rho(E) \leqslant \text{const.}\, p(E - 2\nu)$$

for all real E.

Corollary 2.2. Let $d\lambda$ be the Gaussian distribution defined in (1.4). If $|\eta| > 0$ and $|E| \gg 1$ then the expansion (2.5) converges uniformly, and $\rho(E)$ is an analytic function of E, for $|E| \gg 1$ and $|\mathrm{Im}\, E| < (1/\sqrt{2})|\mathrm{Re}\, E|$.

Moreover, for real E,

$$0 < \rho(E) \leqslant \text{const.} \exp\left[-\gamma(E - \sigma_E)^2\right]$$

where $\sigma_E = \epsilon$ if $E < 0$, $\sigma_E = 4\nu + \epsilon$ if $E > 0$, and ϵ is an arbitrarily small, positive number; ($\epsilon = 0$, if $\nu \geqslant 3$).

Theorem 2.3. Let $d\lambda$ be the Gaussian given by (1.4). If $0 < \gamma \ll 1$ (i.e., for high disorder) the expansion (2.5) converges uniformly in $z = E + i\eta$, $|\eta| > 0$. The left side of (2.5) has an analytic continuation in z across the cut to a uniform neighborhood of the real axis. The density of states, $\rho(E)$, is a real analytic function of E.

Remark. One also expects $\rho(E)$ to be analytic when $d\lambda$ is the Gaussian with $\gamma \gg 1$ and $|E| \leqslant$ const. However, our methods do not apply in this region.

Proof of Theorem 2.1. In order to establish convergence of the expansion (2.5) for z in a large domain we must estimate the functions

$$I_r(z, d\lambda) \equiv \int d\lambda(V)(2\nu - z + V)^{-r} \tag{2.9}$$

Let $z = E + i\eta$, E real. We define

$$\Gamma_1 = \{V : |\mathrm{Im}\, V| = 0, \mathrm{Re}\, V \leqslant E - 2\nu - l\}$$

and, for $\eta < 0$

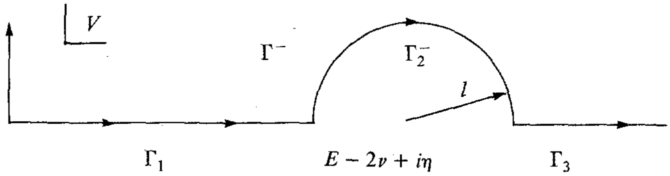
$$\Gamma_2^- = \{V : |V - E + 2\nu| = l, 0 < \arg(V - E + 2\nu) < \pi\}$$

while, for $\eta > 0$,

$$\Gamma_2^+ = \{ V : |V - E + 2\nu| = l, \ -\pi < \arg(V - E + 2\nu) < 0 \}$$

Finally,

$$\begin{aligned} \Gamma_3 &= \{ V : |\mathrm{Im}\, V| = 0, \ \mathrm{Re}\, V \geqslant E - 2\nu + l \} \\ \Gamma^\pm &= \Gamma_1 \cup \Gamma_2^\pm \cup \Gamma_3 \end{aligned} \tag{2.10}$$



Let $d\lambda(V) = w(V)\,dV$, where w is analytic in a neighborhood of the domain bounded by the real axis and $\Gamma^\pm$. We then have

$$I_r(z, d\lambda) = \int_{\Gamma^\pm} (2\nu - z + V)^{-r} w(V)\, dV \tag{2.11}$$

for $z = E + i\eta$, $\eta \gtrless 0$.

Since, for $|\eta| \neq 0$,

$$\begin{aligned} \left| \int_{\Gamma_1 \cup \Gamma_3} d\lambda(V)(2\nu - z + V)^{-r} \right| &\leqslant l^{-r} \int d\lambda(V) = l^{-r} \\ \left| \int_{\Gamma_2^\pm} d\lambda(V)(2\nu - z + V)^{-r} \right| &\leqslant \pi l^{-r+1} \max_{V \in \Gamma_2^\pm} |w(V)| \end{aligned} \tag{2.12}$$

we have

$$|I_r(z, d\lambda)| \leqslant l^{-r} \Big[1 + \pi l \max_{V \in \Gamma_2^\pm} |w(V)| \Big] \tag{2.13}$$

If w satisfies (2.6)–(2.8) we may choose

$$l = 2\nu + \epsilon$$

and obtain

$$|I_r(z, d\lambda)| \leqslant (2\nu + \epsilon)^{-r} \big[1 + \pi(2\nu + \epsilon) p(E - 2\nu) \big] \tag{2.14}$$

From (2.14) and (2.8) it now follows that, given $0 \leqslant \delta < \epsilon$, there exists some finite constant E_δ such that, for $|E| \geqslant E_\delta$,

$$|I_r(z, d\lambda)| \leqslant (2\nu + \delta)^{-r} \tag{2.15}$$

Inserting (2.15) into (2.5) we obtain

$$\begin{aligned}
&\left|\int d\bar{\lambda}(V)\langle x|R_V(z)|y\rangle\right| \\
&\quad\leqslant \sum_{\omega : x\to y}\ \prod_{j\in\mathbb{Z}^\nu} |I_{n_j(\omega)}(z,d\lambda)| \\
&\quad\leqslant \sum_{\omega : x\to y}\ \prod_{j\in\mathbb{Z}^\nu} (2\nu+\delta)^{-n_j(\omega)} \\
&\quad= (-\Delta+\delta)^{-1}(x-y)
\end{aligned} \tag{2.16}$$

for $z = E + i\eta$, $|\eta| \neq 0$, $0 < \delta < \epsilon$ and $|E| \geqslant E_\delta$. The right-hand side of (2.16) decays exponentially in $|x-y|$. This completes the proof of part (1) of Theorem 2.1.

We now turn to the proof of part (2). By Proposition 1, (1.5) and (1.6),

$$\rho(E) = \lim_{\eta\downarrow 0}\int d\bar{\lambda}(V)\left[\langle x|R_V(E-i\eta)|x\rangle - \langle x|R_V(E+i\eta)|x\rangle\right] \tag{2.17}$$

For $z = E - i\eta$, $|\eta| > 2\nu$, (2.3) yields

$$\begin{aligned}
&\langle x|R_V(z)|x\rangle - \langle x|R_V(\bar{z})|x\rangle \\
&\quad= \sum_{\omega : x\to x}\left\{\prod_j\left[2\nu - z + V(j)\right]^{-n_j(\omega)} - \prod_j\left[2\nu-\bar{z}+V(j)\right]^{-n_j(\omega)}\right\}
\end{aligned} \tag{2.18}$$

For each random walk ω, let $j_1 = j_1(\omega), \dots, j_M = j_{M(\omega)}(\omega)$ be the set of sites visited by ω, ordered in an arbitrary way. We now apply the identity

$$\prod_{\alpha=1}^{M} a_\alpha^+ - \prod_{\alpha=1}^{M} a_\alpha^- = \sum_{k=0}^{M-1}\prod_{\alpha=1}^{k} a_\alpha^+ (a_{k+1}^+ - a_{k+1}^-)\prod_{\alpha=k+2}^{M} a_\alpha^-$$

with $a_\alpha^+ = [2\nu - z + V(j_\alpha)]^{-n_{j_\alpha}(\omega)}$, $a_\alpha^- = \overline{a_\alpha^+}$, to the right-hand side of (2.18). After integration with $d\bar{\lambda}$ this yields

$$\begin{aligned}
&\int d\bar{\lambda}(V)\left\{\langle x|R_V(z)|x\rangle - \langle x|R_V(\bar{z})|x\rangle\right\} \\
&\quad= \sum_{\omega : x\to x}\ \sum_{k=0}^{M(\omega)-1}\ \prod_{\alpha=1}^{k} I_{n_{j_\alpha}(\omega)}(z,d\lambda) \\
&\qquad\times\left(I_{n_{j_{k+1}}(\omega)}(z,d\lambda) - I_{n_{j_{k+1}}(\omega)}(\bar{z},d\lambda)\right)\prod_{\alpha=k+2}^{M(\omega)} I_{n_{j_\alpha}(\omega)}(\bar{z},d\lambda)
\end{aligned} \tag{2.19}$$

We now establish the following properties of $I_r(z,d\lambda)$, requiring (2.6)–(2.8):

(a) $I_r(E \pm i\eta, d\lambda)$, $\eta > 0$, has an analytic continuation in E to the domain

$$\left\{ E : \begin{matrix} \operatorname{Im} E > -2\nu - \epsilon \\ \operatorname{Im} E < 2\nu + \epsilon \end{matrix} \right\} \tag{2.20}$$

and

$$|I_r(E \pm i\eta, d\lambda)|$$
$$\leqslant (2\nu + \epsilon)^{-r} + \operatorname{dist}\left(E \pm i\eta, \Gamma_2^{\pm}\right)^{-r} \pi(2\nu + \epsilon) \max_{V \in \Gamma_2^{\pm}} |w(V)| \tag{2.21}$$

(b) $$D_r(E, d\lambda) \equiv \lim_{\eta \downarrow 0} \{ I_r(E - i\eta, d\lambda) - I_r(E + i\eta, d\lambda) \} \tag{2.22}$$

has an analytic continuation in E to the strip

$$\{ E : |\operatorname{Im} E| < 2\nu + \epsilon \} \tag{2.23}$$

and

$$|D_r(E, d\lambda)| \leqslant \operatorname{dist}(E, \Gamma_2)^{-r} 2\pi(2\nu + \epsilon) \max_{V \in \Gamma_2} |w(V)| \tag{2.24}$$

where $\Gamma_2 = \Gamma_2^+ \cup \Gamma_2^-$.

Proof of (a). For E real and $\eta > 0$ it follows from (2.6), (2.7) that

$$I_r(E - i\eta, d\lambda) = \int_{\Gamma^-} d\lambda(V)(2\nu - E + i\eta + V)^{-r} \tag{2.25}$$

where Γ^- is defined in (2.11), and we set $l = 2\nu + \epsilon$. The right side of (2.25) obviously has an analytic continuation in E to the domain defined in (2.20). The bounds on $|I_r(E - i\eta, d\lambda)|$ on that domain are proven as in (2.12), (2.13). The proof of (a) for $I_r(E + i\eta, d\lambda)$ is similar.

In order to prove (b), we note that

$$\lim_{\eta \downarrow 0} \int_{\Gamma_1 \cup \Gamma_3} d\lambda(V) \left\{ (2\nu - E + i\eta + V)^{-r} - (2\nu - E - i\eta + V)^{-r} \right\} = 0$$

where Γ_1, Γ_3 are as in (2.10). Hence

$$D_r(E, d\lambda) = \oint_{\Gamma_2} d\lambda(V)(2\nu - E + V)^{-r} \tag{2.26}$$

from which (b) follows immediately.

Next, we use (a) and (b) to prove absolute convergence of the expansion (2.19) for z in a suitable domain in the complex plane: If $z = E - i\eta$, $\eta > 0$ and $|\operatorname{Im} E| < \epsilon/2$ then one can use (2.21) to establish absolute convergence of the right side of (2.19), provided $|\operatorname{Re} E|$ is large enough; see (2.8). In particular,

$$|I_r(E \pm i\eta, d\lambda)| \leqslant (2\nu + \epsilon/2)^{-r}\left[1 + \pi(2\nu + \epsilon) p(\operatorname{Re} E - 2\nu) \right]$$
$$|D_r(E, d\lambda)| \leqslant (2\nu + \epsilon/2)^{-r} 2\pi(2\nu + \epsilon) p(\operatorname{Re} E - 2\nu)$$

if $|\operatorname{Im} E| < \epsilon/2$ and $\eta > 0$. Thus, by (2.19),

$$\begin{aligned}&\left|\lim_{\eta\downarrow 0}\int d\bar{\lambda}(V)\{\langle x|R_V(E-i\eta)|x\rangle - \langle x|R_V(E+i\eta)|x\rangle\}\right|\\ &\quad\leqslant \sum_{\omega\,:\,x\to x}|\omega|(2\nu+\delta)^{-|\omega|}2\pi(2\nu+\epsilon)p(\operatorname{Re}E-2\nu)\\ &\quad\leqslant \text{const. } p(\operatorname{Re}E-2\nu)\end{aligned}\tag{2.27}$$

where δ is chosen to lie in $(0,\epsilon/2)$, and $|\operatorname{Re} E|$ is so large that

$$(2\nu+\epsilon/2)^{-1}\left[1+\pi(2\nu+\epsilon)p(\operatorname{Re}E-2\nu)\right]\leqslant(2\nu+\delta)^{-1}$$

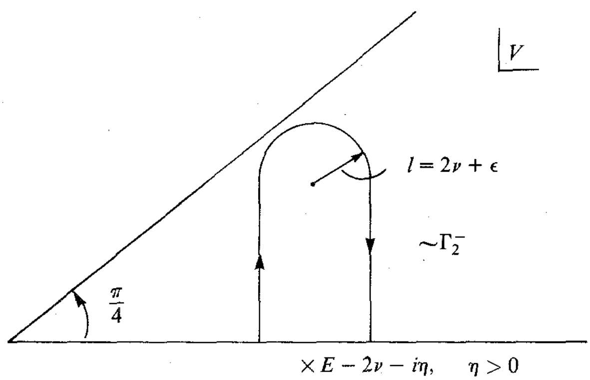
Moreover, $|\omega|-1$ is the total number of jumps a walk ω makes. These estimates (along with Wegner's results[14]) complete the proof of part (2) of Theorem 2.1. ■

We now turn to the proof of Corollary 2.2. The Gaussian distribution,

$$d\lambda(V) = w(V)\,dV$$

$$w(V) = (\gamma/2\pi)^{1/2}\exp\left[-(\gamma/2)V^2\right]$$

satisfies properties (2.6), (2.7), and (2.8), for arbitrary values of $\epsilon > 0$. This proves the first part of Corollary 2.2. In claims (a) and (b) above, we choose $\Gamma_2^{\pm}$ as indicated below:



Inserting the upper bound

$$|w(V)|\leqslant(\gamma/2\pi)^{1/2}\exp\left[-(\gamma/2)\left\{(\operatorname{Re}V)^2-(\operatorname{Im}V)^2\right\}\right]$$

into estimates (2.21) and (2.24) we obtain convergence of the expansion (2.19), and hence the analyticity properties of $\rho(E)$ claimed in Corollary

2.2, provided $|\mathrm{Im}\, E| < \alpha|\mathrm{Re}\, E|$, $\alpha < 1$, and $|\mathrm{Re}\, E|$ is large enough depending on α.

If E is kept real we obtain the upper bound on $\rho(E)$. ■

Proof of Theorem 2.3. If in (2.12), (2.13), (2.25), and (2.26) we choose

$$l = \kappa\gamma^{-1/2}$$

for some constant $\kappa > 0$ (in which one optimizes subsequently), we obtain

$$|I_r(z, d\lambda)| \leqslant (\gamma^{1/2}/\kappa)^r\left[1 + \mathrm{const}\,\kappa e^{\kappa/2}\right] \tag{2.13'}$$

where $z = E + i\eta$, $|\eta| > 0$, and

$$|D_r(E, d\lambda)| \leqslant \mathrm{const}(\gamma^{1/2}/\kappa)^r e^{\kappa/2} \tag{2.24'}$$

uniformly in $E \in \mathbb{R}$. Similar estimates which are uniform in $\mathrm{Re}\, E$ hold when $|\mathrm{Im}\, E| \leqslant$ const. Thus, for $0 < \gamma \ll 1$, Theorem 2.3 follows. ■

3. THE REPLICA METHOD

3.1. Introductory Remarks

As mentioned in the Introduction, the replica method is a means of calculating, e.g.,

$$\int d\bar{\lambda}(V)\langle x|R_V(z)|y\rangle$$

and other quantities of interest, in terms of Green's functions of N-component scalar field theories in the limit where N tends to 0; see, e.g., Ref. 11 for a summary of that method. Here we propose to justify the replica method in the region where $|E| = |\mathrm{Re}\, z|$ is large or when there is large disorder, i.e., in the region where localized states are expected.

The plan of this section is as follows: We first prove the first equation in (1.8), i.e.,

$$\begin{aligned}\int d\bar{\lambda}(V)&\langle x|R_V(z)|y\rangle\\ &= \lim_{N\to 0} iZ_N^{-1}\int d\bar{\lambda}(V)\int d\phi\,\phi^1(x)\phi^1(y)\exp\left[-\frac{i}{2}(\phi, (-\Delta + V - z)\phi)\right]\end{aligned} \tag{3.1}$$

where $z = E + i\eta$, $\eta > 0$. The quotient on the right side of (3.1) must be understood as the thermodynamic limit of quotients in finite volume which are defined as follows:

Let Δ^Λ be the finite difference Laplacian in a finite box Λ with some boundary conditions (e.g., periodic or Dirichlet) imposed at the boundary $\partial\Lambda$. Let

$$\phi_\Lambda = \{\phi(j)\}_{j\in\Lambda}, \qquad d\phi_\Lambda = \prod_{j\in\Lambda}\prod_{\alpha=1}^{N} d\phi^\alpha(j)$$

and $d\lambda_\Lambda(V) = \prod_{j\in\Lambda} d\lambda(V(j))$.

Finally, let $A(\phi)$ be some function depending on only finitely many $\phi(j)$'s. We define the unnormalized expectation

$$\begin{aligned}[A(\phi)]_{N,\Lambda} &= \int d\lambda_\Lambda(V)\int d\phi_\Lambda A(\phi)\\ &\quad\times \exp\left[-\frac{i}{2}\left(\phi_\Lambda, (-\Delta^\Lambda + V - z)\phi_\Lambda\right)\right]\end{aligned} \tag{3.2}$$

and

$$Z_{N,\Lambda} = [1]_{N,\Lambda} \tag{3.3}$$

The normalized expectation in finite volume is defined by

$$\langle A(\phi)\rangle_{N,\Lambda} = Z_{N,\Lambda}^{-1}[A(\phi)]_{N,\Lambda} \tag{3.4}$$

We propose to show that $\langle A(\phi)\rangle_{N,\Lambda}$ has an analytic interpolation in N valid in some neighborhood of $[0,\infty)$, for arbitrary Λ. Then we shall show that for sufficiently large $|\mathrm{Re}\, z|$ or high disorder and for N in some Λ-independent complex domain containing the origin $N = 0$,

$$\langle A(\phi)\rangle_N = \lim_{\Lambda\nearrow\mathbb{Z}^d}\langle \mathbf{A}(\phi)\rangle_{N,\Lambda} \tag{3.5}$$

exists and is analytic in N inside that domain. Subsequently we prove that

$$\lim_{N\to 0} i\langle\phi^1(x)\phi^1(y)\rangle_N = \int d\bar\lambda(V)\langle x|R_V(z)|y\rangle \tag{3.6}$$

Finally, for

$$d\lambda(V) = (\gamma/2\pi)^{1/2}\exp\left[-\frac{\gamma}{2}V^2\right]dV$$

we shall relate $\langle A(\phi)\rangle_N$, $N = 1,2,3,\ldots,$ to a Green's function of the $g|\phi|^4$ theory, continued analytically in the mass ($m^2 = -z$) and the coupling constant ($g \propto -\gamma^{-1}$!)

The methods used below are not restricted to quantities like $\int d\bar\lambda(V)$ $\langle x|R_V(z)|y\rangle$ or $\rho(E)$, but can also be used to justify the replica method in calculations of expressions related to the conductivity or the diffusion coefficient which can be expressed in terms of integrals of the form

$$\sum_{x,y,x',y'}\int d\bar\lambda(V) g(x,y,x',y')\langle x|R_V(z)|y\rangle\langle x'|R_V(z)|y'\rangle \tag{3.7}$$

Such integrals require two sets of replicas, because there are two distinct "mass parameters," z and $\bar{z}$. Unfortunately, our methods do not permit us to analyze the behavior of integrals like (3.7) with $z = E + i\eta$, in the limit when η tends to 0. For this reason we shall limit our discussion to the replica method for quantities related to $\rho(E)$.

3.2. Replicas in Finite Volume

We now analyze the integrals on the right side of equations (3.2) and (3.3). If $z = E + i\eta$, $\eta > 0$,

$$Z_{N,\Lambda} = \int d\lambda_\Lambda(V) \int d\phi_\Lambda \exp - \left[\frac{i}{2} \left(\phi_\Lambda, (-\Delta^\Lambda + V - z)\phi_\Lambda \right) \right]$$
$$= \int d\lambda_\Lambda(V) \det\left[\frac{i}{2\pi} (-\Delta^\Lambda + V - z) \right]^{-N/2} \tag{3.8}$$

and

$$\left[\phi^1(x)\phi^1(y) \right]_{N,\Lambda} = \int d\lambda_\Lambda(V) \int d\phi_\Lambda \phi^1(x)\phi^1(y)$$
$$\times \exp\left[-\frac{i}{2} \left(\phi_\Lambda, (-\Delta^\Lambda + V - z)\phi_\Lambda \right) \right]$$
$$= -i \int d\lambda_\Lambda(V)(-\Delta^\Lambda + V - z)^{-1}(x, y)$$
$$\times \det\left[\frac{i}{2\pi} (-\Delta^\Lambda + V - z) \right]^{-N/2} \tag{3.9}$$

Notice that, for $\operatorname{Im} z > 0$ and finite Λ, all integrals in (3.8) and (3.9) exist. In order to analyze the properties of these integrals, we expand $(-\Delta^\Lambda + V - z)^{-1}$ and $\det[(i/2\pi)(-\Delta^\Lambda + V - z)]^{-N/2}$ in random walks, as in Section 2 and Ref. 12. Thus, for periodic or Dirchlet boundary conditions at $\partial\Lambda$,

$$(-\Delta^\Lambda + V - z)^{-1}(x, y) = \sum_{\omega : x \to y} \prod_{j \in \Lambda} D_j(z)^{-n_j(\omega)} \tag{3.10}$$

With $D_j(z) = 2\nu - z + V(j)$; see (2.1)–(2.3). Moreover,

$$\det\left[\frac{i}{2\pi} (-\Delta^\Lambda + V - z) \right]^{-N/2}$$
$$= \exp\left[-\frac{N}{2} \operatorname{tr} \ln\left\{ \frac{i}{2\pi} (-\Delta^\Lambda + V - z) \right\} \right]$$
$$= \exp\left[-\frac{N}{2} \operatorname{tr} \ln\left(\frac{i}{2\pi} D \right) \right]$$
$$\times \exp\left[-\frac{N}{2} \operatorname{tr} \ln(1 - D^{-1}P^\Lambda) \right]$$

where P^{Λ} is the off-diagonal part of Δ^{Λ}. Using the series expansion for the logarithm we obtain

$$\det\left[\frac{i}{2\pi}(-\Delta^{\Lambda}+V-z)\right]^{-N/2}$$
$$= C_{N,\Lambda}\prod_{j\in\Lambda} D_j(z)^{-N/2}$$
$$\times\exp\left\{\frac{N}{2}\sum_{j\in\Lambda}\sum_{\omega\,:\,j\to j}|\omega|^{-1}\prod_{j\in\Lambda} D_j(z)^{-n_j(\omega)}\right\} \tag{3.11}$$

where

$$C_{N,\Lambda}=(-2\pi i)^{(N/2)|\Lambda|},\qquad |\omega|=\sum_{j\in\Lambda} n_j(\omega)$$

For finite Λ, the expansions (3.10) and (3.11) converge uniformly in V, provided $\operatorname{Im} z > 2\nu$. Let $\Omega=\{\omega_1,\omega_2,\ldots\}$ be an ordered set of not necessarily distinct random loops, and let $|\Omega|$ denote the number of random loops in Ω. Let $z_0(\Omega)\equiv(N/2)^{|\Omega|}\prod_{\omega\in\Omega}|\omega|^{-1}$. If we expand the exponential on the right side of (3.11) and insert the result back into equations (3.8) and (3.9) we obtain

$$Z_{N,\Lambda}=C_{N,\Lambda}\left\{\sum_{\Omega} z_0(\Omega)(|\Omega|!)^{-1}\int d\lambda_\Lambda(V)\prod_{j\in\Lambda} D_j(z)^{-[N/2+n_j(\Omega)]}\right\} \tag{3.12}$$

where $n_j(\Omega)=\sum_{\omega\in\Omega} n_j(\omega)$, and

$$\left[\phi^1(x)\phi^1(y)\right]_{N,\Lambda}=C_{N,\Lambda}\left\{\sum_{\omega\,:\,x\to y}\sum_{\Omega} z_0(\Omega)(|\Omega|!)^{-1}\right.$$
$$\left.\times\int d\lambda_\Lambda(V)\prod_{j\in\Lambda} D_j(z)^{-[N/2+n_j(\Omega)+n_j(\omega)]}\right\} \tag{3.13}$$

We observe that the integrals on the right side of (3.12) and (3.13) factorize. Recalling the definition (2.12) of the functions $I_r(z,d\lambda)$ we obtain

$$Z_{N,\Lambda}=C_{N,\Lambda}\sum_{\Omega} z_0(\Omega)(|\Omega|!)^{-1}\prod_{j\in\Lambda} I_{N/2+n_j(\Omega)}(z,d\lambda) \tag{3.14}$$

and

$$\left[\phi^1(x)\phi^1(y)\right]_{N,\Lambda}=C_{N,\Lambda}\sum_{\omega\,:\,x\to y}\sum_{\Omega} z_0(\Omega)(|\Omega|!)^{-1}$$
$$\times\prod_{j\in\Lambda} I_{N/2+n_j(\Omega)+n_j(\omega)}(z,d\lambda) \tag{3.15}$$

We now define

$$\begin{aligned}\tilde{Z}_{N,\Lambda}&=I_{N/2}(z,d\lambda)^{-|\Lambda|}C_{N,\Lambda}^{-1}Z_{N,\Lambda}\\ [(\cdot)]^{\sim}_{N,\Lambda}&=I_{N/2}(z,d\lambda)^{-|\Lambda|}C_{N,\Lambda}^{-1}[(\cdot)]_{N,\Lambda}\end{aligned} \tag{3.16}$$

and

$$W(\Omega) = \prod_{j \in \Lambda} \left(I_{N/2+n_j(\Omega)}(z, d\lambda) / I_{N/2}(z, d\lambda) \right) \tag{3.17}$$

Dividing (3.14) and (3.15) by $I_{N/2}(z, d\lambda)^{|\Lambda|}$, we obtain

$$\tilde{Z}_{N,\Lambda} = \sum_{\Omega} z_0(\Omega)(|\Omega|!)^{-1} W(\Omega) \tag{3.14'}$$

$$\left[\phi^1(x)\phi^1(y)\right]^{\sim}_{N,\Lambda} = \sum_{\omega\,:\,x \to y} \sum_{\Omega} z_0(\Omega)(|\Omega|!)^{-1} W(\Omega \cup \omega) \tag{3.15'}$$

Our next task is to study the convergence of (3.14′) and (3.15′) uniformly in Λ. This is accomplished with the help of a cluster expansion. We use the polymer method described in Refs. 7, 9, and 13. First we note that a term on the right side of (3.14′) or (3.15′) indexed by a family

$$\Omega = \{\omega_1, \ldots, \omega_m\}, \qquad m = 1, 2, 3, \ldots, \text{ of random loops}$$

does not depend on the ordering of $\{\omega_1, \ldots, \omega_m\}$. Thus we may resum over families Ω which only differ in their ordering. The resulting expansions of $\tilde{Z}_{N,\Lambda}$ and $[\phi^1(x)\phi^1(x)]^{\sim}_{N,\Lambda}$ are indexed by "multi-indices",[(13)] i.e, functions from the class Γ of all random loops to $\mathbb{N}$. In accordance with Ref. 13 we shall call these multi-indices g-sets (for "generalized sets"), and we shall denote them again by Ω. Given a g-set Ω and a random loop ω, let $\nu(\Omega, \omega)$ be the total number of copies of ω appearing in Ω. We set

$$[\Omega]! = \prod_{\omega \in \Gamma} \nu(\Omega, \omega)! \tag{3.18}$$

We may now rewrite (3.14′) and (3.15′) in terms of sums over g-sets of random loops, replacing $|\Omega|!$ by $[\Omega]!$. Next, we observe that each term indexed by some g-set in these new expansions factorizes in a product of terms labeled by *connected* g-sets[5] which are called "*polymers*" and denoted by Ω^c. For, if $\Omega_1 \cap \Omega_2 = \emptyset$ then

$$W(\Omega_1 \cup \Omega_2) = W(\Omega_1) W(\Omega_2), \qquad z_0(\Omega_1 \cup \Omega_2) = z_0(\Omega_1) z_0(\Omega_2) \tag{3.19}$$

as follows from (3.17). Two polymers, Ω_1^c, Ω_2^c are said to be *compatible* iff $\Omega_1^c \cup \Omega_2^c$ is not a polymer, i.e., not connected. Otherwise they are said to be *incompatible*. If Ω is a g-set of random loops and ω a random walk starting at x and ending at u such that $\Omega \cup \omega$ is connected, then

$$\Omega_{xu}^c \equiv \Omega \cup \omega$$

is called an $x \to u$ polymer. Let $|\Omega^c|, [\Omega^c]!, W(\Omega^c)$ be defined as above (with

[5] A g-set is connected if it consists of random loops $\omega_1, \ldots, \omega_m$—not necessarily distinct—such that any point on ω_i can be joined to any point on ω_j by a sequence of nearest-neighbor jumps in $\bigcup_{k=1}^m \omega_k$, for all i, j.

$\Omega \to \Omega^c$). With each polymer Ω^c we associate an *activity*, $z(\Omega^c)$, defined by

$$z(\Omega^c) = z_0(\Omega^c)([\Omega^c]!)^{-1} W(\Omega^c) \tag{3.20}$$

Clearly, all these notions are also defined for $x \to u$ polymers. The expansions (3.14′), (3.15′) can now be rewritten as sums of products of activities associated with compatible polymers. Compatibility may be viewed as arising from a hard core exclusion between different polymers. We define

$$g(\Omega_1^c, \Omega_2^c) = \begin{cases} 0, & \text{if } \Omega_1^c \text{ and } \Omega_2^c \text{ are compatible} \\ -1, & \text{otherwise} \end{cases} \tag{3.21}$$

Then the expansions (3.14′) and (3.15′) take the form

$$\tilde{Z}_{N,\Lambda} = \sum_{r=0}^{\infty} \sum_{\{\Omega_1^c, \ldots, \Omega_r^c\}} \prod_{k=1}^{r} z(\Omega_k^c) \prod_{1 \leqslant k < k' \leqslant r} [1 + g(\Omega_k^c, \Omega_{k'}^c)] \tag{3.22}$$

and

$$\langle \phi^1(x) \phi^1(u) \rangle_{N,\Lambda} = \tilde{Z}_{N,\Lambda}^{-1} \left\{ \sum_{r=0}^{\infty} \sum_{\{\Omega_1^c, \ldots, \Omega_r^c\}} \prod_{k=1}^{r} z(\Omega_k^c) \right.$$
$$\left. \times \prod_{1 \leqslant k < k' \leqslant r} [1 + g(\Omega_k^c, \Omega_{k'}^c)] \right\} \tag{3.23}$$

where $\{\Omega_1^c, \ldots, \Omega_r^c\}$ are *unordered* sets of polymers. In (3.23), one polymer, Ω_1^c, is an $x \to u$ polymer. The factor

$$\prod_{1 \leqslant k < k' \leqslant r} [1 + g(\Omega_k^c, \Omega_{k'}^c)]$$

can be interpreted as the Boltzmann factor of r polymers indexed by the g-sets $\Omega_1^c, \ldots, \Omega_r^c$ and interacting via hard core exclusion, with $z(\Omega^c)$ playing the role of a chemical activity of Ω^c. It is possible to reduce the expansion (3.23) to an expansion of the form (3.22)[9,13]:

$$\langle \phi^1(x) \phi^1(u) \rangle_{N,\Lambda} = \frac{d}{dt} \log \left\{ \tilde{Z}_{N,\Lambda} + t[\phi^1(x)\phi^1(u)]_{N,\Lambda}^{\sim} \right\} \Big|_{t=0} \tag{3.24}$$

The expression within $\{ \cdots \}$ is a modified partition function.

3.3. Cluster Expansion and Thermodynamic Limit

Here we use the polymer method, described in Refs. 7–9 and 15 to prove convergence. In order to control the logarithms of polymer expansions we require two basic properties (A and B below) of polymers.[9,13] We define the length $l(\Omega^c)$, of a polymer Ω^c by

$$l(\Omega^c) = \sum_{\omega \in \Omega^c} |\omega|$$

If Ω^c is an $x \to u$ polymer, $\Omega^c_{xu} = \Omega \cup \omega$, we set

$$l(\Omega^c_{xu}) = |\omega| + \sum_{\omega' \in \Omega} |\omega'|$$

Convergence is ensured if the following properties hold:

Property A. Let Ω^c_0 be a polymer of length l_0. The total number of polymers, Ω^c, of length l which are incompatible with Ω^c_0 is bounded by $l_0 K_1^l$, where K_1 is some constant (which turns out to be proportional to the dimension ν of the lattice).

Property B. The activity, $z(\Omega^c)$, of each polymer Ω^c is bounded by $K_2^{l(\Omega^c)}$, where K_2 is a constant depending on ν.

The issue is now to verify Properties A and B for our case. Before we go into this, we recall the consequences of convergence of the cluster expansion, guaranteed by Properties A and B.

Let X denote a g-set of polymers. If X contains an $x \to u$ polymer, Ω^c_{xu}, we write $X : x \to u$. With each g-set X we associate a graph $G(X)$ whose vertices are the polymers in X. Two vertices of $G(X)$ are joined by a line iff the corresponding polymers are incompatible. The number of lines (pairs of incompatible polymers) in a graph, G, is denoted by $L(G)$. Let $\nu(X, \Omega^c)$ be the number of copies of polymer Ω^c in a g-set X,

$$[X]! = \prod_{\Omega^c} \nu(X, \Omega^c)!$$

and

$$z(X) = \prod_{\Omega^c} z(\Omega^c)^{\nu(X, \Omega^c)}$$

The result of the cluster expansion is summarized as follows:

Lemma 3.1[9,13]. We have

$$\log \tilde{Z}_{N,\Lambda} = \sum_X \frac{1}{[X]!} \phi^T(X) z(X) \tag{3.25}$$

$$\langle \phi^1_x \phi^1_u \rangle_{N,\Lambda} = \sum_{X : x \to u} \frac{1}{[X]!} \phi^T(X) z(X) \tag{3.26}$$

where all g-sets X only contain sites in Λ, and

$$\phi^T(X) = \sum_{G \subset G(X)} (-1)^{L(G)} \tag{3.27}$$

the sum ranging over *all connected* subgraphs G of $G(X)$ containing *all* the vertices of $G(X)$.

Remark. The coefficients $\phi^T(X)$ have a purely combinatorial character. They vanish, unless $G(X)$ is connected (i.e., X is connected). In Refs. 8 and 9 the following upper bound on $|\phi^T(X)|$ has been established.

Lemma 3.2. Under the assumption that Property A holds

$$|\phi^T(X)| \leqslant [X]!K_3^{l(X)} \tag{3.28}$$

where $l(X) = \sum_{\Omega^c} \nu(X, \Omega^c) l(\Omega^c)$, and K_3 is a geometric constant depending on ν.

Remark. This result is usually presented in two portions: one first establishes a general estimate on the right-hand side of (3.27) and then proves an upper bound on that estimate, exploiting the lattice structure and Property A. We do not repeat the proofs of Lemmas 3.1 and 3.2. The reader may consult the clear presentations in Refs. 15 and 9, and for an application similar to the present one, Ref. 13.

We now proceed to verify Properties A and B for our case.

To verify Property A we use the construction described in Ref. 13. With each polymer Ω^c we associate a walk $\overline{\Omega}^c$ which completely follows all the walks in Ω^c and two appropriately defined subsets $S(\Omega^c)$ and $E(\Omega^c)$ made of nearest-neighbor steps of the random walks in Ω^c (including steps of the open walk $x \to u$ if Ω^c is an $x \to u$ polymer). Next, one finds that, given a random walk ω and two subsets S and E of jumps, there exists at most one polymer Ω^c such that $\overline{\Omega}^c = \omega$, $S(\Omega^c) = S$ and $E(\Omega^c) = E$. Since there are no more than $2^{l(\omega)} \times 2^{l(\omega)} = 4^{l(\omega)}$ possible subsets S and E for a given ω and since there are no more than $l_0(2\nu)^l$ random walks of length l incompatible with Ω_0^c, the number of polymers, Ω^c, of length l incompatible with Ω_0^c is bounded by

$$l_0 4^{l(\Omega^c)}(2\nu)^l \leqslant l_0(8\nu)^l$$

which is the required estimate.

We now come to the proof of Property B. In order to prove the bound on $z(\Omega^c)$ we use the estimates on $I_r(z, d\lambda)$, proved in Section 2. for the case of the Gaussian distribution (1.4). Indeed, given $\delta > 0$, there exists some finite constant E_δ such that, for $|E| \geqslant E_\delta$ we have [see (2.15)]

$$|I_r(z, d\lambda)| \leqslant (2\nu + \delta)^{-r} \tag{3.29}$$

On the other hand, for the case of high disorder $0 < \gamma \ll 1$, we have from (2.13′)

$$|I_r(z, d\lambda) \leqslant \text{const}\left(\sqrt{\gamma}\right)^r \tag{3.30}$$

Certainly, for complex values of N near $N = 0$, the integral $I_N(z, d\lambda)$ is bounded from below by a constant, uniformly in N. The required estimate for $z(\Omega^c)$ follows from the relations (3.17) and (3.20), using (3.29) for large energies or (3.30) for the case of large disorder. As stated above, the convergence of the expansions (3.25) and (3.26) follows now from Properties A and B.

Finally, we wish to return to Eq. (1.8) and explain the connection of the replica method, as studied in this paper, with the $N \to 0$ limit of the $O(N)$-invariant $g|\phi|^4$ model, with negative coupling constant g.

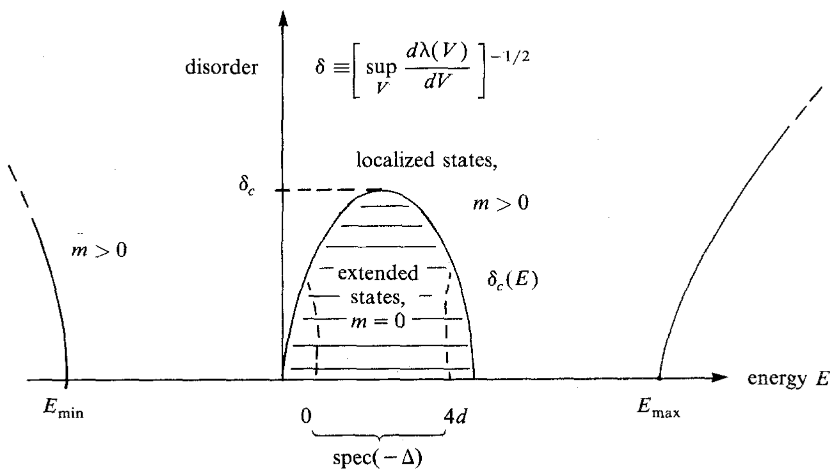
The expression in the middle of Eq. (1.8) can be interpreted as the $N \to 0$ limit of the $g|\phi|^4$ model, with (complex) mass $m^2 = -z = -E - i\eta$ and negative coupling constant $g = -(8\gamma)^{-1}$. If we choose the energy E to be large and negative then the modulus of m^2 is large, and we can interpret the right-hand side of (1.8), before taking the $N \to 0$ limit, as an analytic continuation of the $g|\phi|^4$ model from positive values of the coupling constant $g = (8\gamma)^{-1}$ to the negative axis of the complex g plane. In fact, using the cluster expansion, one can easily show that the correlation functions of the $g|\phi|^4$ model can be analytically continued in the complex g plane to the second Riemann sheet, up to a phase $3\pi/2 - \epsilon$, with ϵ small.

4. CONJECTURES AND OPEN PROBLEMS

4.1. On the Critical Exponent ν of the Localization Length

In this section d denotes the number of dimensions, whereas ν is reserved for the critical exponent of the localization length to be defined below.

As is quite well known, one expects that, in three or more dimensions, the properties of the dynamics determined by a random Schrödinger operator, H_V (as defined in Section 1), are described by the following diagram:



Here $E_{\min} \equiv E_{\min}(\delta) = \min\{V: V \in \operatorname{supp} d\lambda(V)\}$, $E_{\max} \equiv E_{\max}(\delta) = 4d + \max\{V: V \in \operatorname{supp} d\lambda(V)\}$, and $m \equiv m(E,\delta) \equiv \xi(E,\delta)^{-1}$ is the inverse *localization length*.[1–4] For fixed disorder, δ, the *mobility edges*, E_c and $\bar{E}_c$, are the roots of the equation $\delta_c(E) = \delta$.

Let $E_i < E_c$ be an eigenvalue for H_V, and let $\psi_i(x)$ be the corresponding eigenvector. (The case $E_i > \bar{E}_c$ is similar and will not be discussed separately. Dense point spectrum near the band tails has been recently proven to exist in Ref. 4 for a fairly large class of distributions $d\lambda$.) The function $\psi_i(x)$ has exponential decay,

$$\begin{aligned} &|\psi_i(x)| \leqslant \operatorname{const} e^{-m|x|}, \qquad \text{with} \\ &m \equiv \lim_{|x|\to\infty} -\frac{1}{|x|}\ln|\psi_i(x)| > 0 \end{aligned} \tag{4.1}$$

It is expected that $m = m(E_i,\delta)$, where the function $m(E,\delta)$ is the exponential decay rate of the Green's function, i.e., of $\langle x|R_V(E+i\eta)|y\rangle$, as $|x-y|\to\infty$, $\eta\to 0$, which we define in this paper as the localization length; see Ref. 4. The Green's function $\langle x|R_V(z)|y\rangle$, x, y in $\mathbb{Z}^d$, $z\in\mathbb{C}$, is defined to be the xy matrix element of the resolvent $(H_V - z)^{-1}$.

We conjecture that

$$m(E,\delta) \sim \operatorname{const}(E_c - E)^{1/2}, \qquad \text{as} \quad E \nearrow E_c \tag{4.2}$$

if the dimension d of the lattice is sufficiently large and $\delta < \delta_c$. Scaling theories of localization predict that, for $d > 2$ and $\delta < \delta_c$,

$$m(E,\delta) \sim \operatorname{const}(E_c - E)^{\nu}, \qquad \text{with} \quad \nu \geqslant 1/2 \tag{4.3}$$

The value d_c^{loc} of the dimension with the property that $\nu = 1/2$, for $d > d_c^{\mathrm{loc}}$, is called the upper *critical dimension*.

In N-body quantum mechanics, bound state wave functions below a threshold, E_s, have exponential decay, with a decay rate, $m(E)$, behaving like

$$m(E) \sim \operatorname{const}(E_s - E)^{1/2} \tag{4.4}$$

(see Ref. 16). For a one-body system with a potential, V, of compact support, $E_s = 0$ and (4.4) follows directly from the equation

$$\psi_i(x) = \left[(\Delta + E_i)^{-1} V\psi_i\right](x) \tag{4.5}$$

and the well-known decay properties of the matrix elements of $(\Delta + E_i)^{-1}$. For a random Schrödinger operator, H_V, an equation of the form of (4.5) is expected to hold approximately in large dimension, with $E_i \to E_i - E_c$ and $V \to V_{\mathrm{eff}}$, where V_{eff} is an average of $V - E_c$ over ever larger blocks, as the distance from the localization region of ψ_i increases. This would yield

$\nu = 1/2$. However, such arguments do not seem to determine d_c^{loc}. We expect that $E_c \to 0$, as $d \to \infty$.

4.2. On the Value of d_c^{loc}

In order to test "mean-field behavior" in the localization problem one may study the average of a product of a Green's function with argument $E + i\eta$ and one with argument $E - i\eta$. Using the random walk expansion one gets

$$
\begin{aligned}
&\langle\langle x|R_V(E-i\eta)|y\rangle;\langle z|R_V(E+i\eta)|w\rangle\rangle \\
&\quad=\int \langle x|R_V(E-i\eta)|y\rangle\langle z|R_V(E+i\eta)|w\rangle\, d\bar{\lambda}(V) \\
&\quad\quad-\left(\int \langle x\,|\,R_V(E-i\eta)|y\rangle\, d\bar{\lambda}(V)\right)\left(\int \langle z|R_V(E+i\eta)|w\rangle\, d\bar{\lambda}(v)\right) \\
&\quad=\sum_{\substack{\omega\cap\omega'\neq\emptyset\\ \omega\,:\,x\to y\\ \omega'\,:\,z\to w}}\Bigg[\prod_{j\in\mathbb{Z}^d}\int d\lambda(V)\frac{1}{(2d+V-E-i\eta)^{n_j(\omega)}(2d+V-E+i\eta)^{n_j(\omega')}} \\
&\quad\quad-\prod_{j\in\mathbb{Z}^d}\int d\lambda(V)\frac{1}{(2d+V-E-i\eta)^{n_j(\omega)}} \\
&\quad\quad\times\int d\lambda(V)\frac{1}{(2d+V-E+i\eta)^{n_j(\omega')}}\Bigg]
\end{aligned}
\tag{4.6}
$$

Each term on the right-hand side of (4.6) is indexed by a pair of walks, ω and ω', which are required to intersect. A site at which ω and ω' intersect corresponds to a singular integral

$$
\int d\lambda(V)\frac{1}{(2d+V-E-i\eta)^n(2d+V-E+i\eta)^m}
\tag{4.7}
$$

with n and m positive and $\eta > 0$ small. The small divisor problem which appears is similar to the one appearing in the study of the dynamics of a quantum mechanical particle in an *attractive* δ-function potential, concentrated along the walk ω. Certainly, this analogy is hampered by the fact that there are complex phases in each term in the expansion of

$$
\langle\langle x|R_V(E-i\eta)|y\rangle;\langle z|R_V(E+i\eta)|w\rangle\rangle
$$

which result in cancellations of singularities. But the interaction between the two walks in (4.6) is strictly local (i.e., nonzero contributions only result from walks ω and ω' which intersect each other), just as in the dynamics of the quantum mechanical particle with a δ-function potential concentrated along a walk ω. Assuming that the walk ω is a simple random walk, we show below that the critical dimension of this quantum mechanical problem is $d_c = 6$. This leads us to argue that the upper critical dimension of the localization problem lies between 4 and 6. Indeed, the local interaction between the two walks in (4.6) contains, as we have seen above, complex phases, so that, on the basis of our analogy with the dynamics of a particle in a δ-function potential concentrated along a simple random walk (or a Brownian path), the critical dimension of the localization problem is expected to lie somewhere between the critical dimension of that quantum mechanical problem with *attractive* and the same problem with *repulsive* δ-function interaction. The critical dimension of the problem with repulsive δ-function potential is well known to be $d_c = 4$.[17] Hence we expect that

$$4 \leqslant d_c^{\mathrm{loc}} \leqslant 6 \tag{4.8}$$

where d_c^{loc} is the (upper) critical dimension of the localization problem.

For the convenience of the reader we give a simple argument for the claim $d_c = 6$ in the case of the Schrödinger equation with attractive δ-function potential concentrated on a Brownian path which underlies (4.8). We emphasize, however, that this result has actually a mathematically rigorous proof (see Ref. 18). We thank B. Simon for providing us with a different proof). In order to understand this result, we use the fact that a Brownian path has Hausdorff dimension two, with probability 1, and consider the Schrödinger operator $-\Delta + V$, where V is a δ function concentrated on a two-dimensional surface Σ, i.e.,

$$V(x) = -\int_\Sigma \delta(x-y)\rho(y)\,d^2y, \qquad x \in \mathbb{R}^d \tag{4.9}$$

where ρ is a nonnegative weight function concentrated on Σ, and the minus sign on the right-hand side of (4.9) accounts for the attractive character of the potential $V(x)$.

We should like to know, under which condition

$$-\Delta + V(x) = -\Delta, \qquad \text{as self-adjoint operators} \tag{4.10}$$

i.e., we want to calculate the dimension d_c such that for $d \geqslant d_c$, $C_0^\infty(\mathbb{R}^d \backslash \Sigma)$ is a core for $-\Delta$. Because $-\Delta + 1$ is an invertible operator with bounded inverse, this will be so iff the set

$$\{(-\Delta + 1)f \,|\, f \in C_0^\infty(\mathbb{R}^d \backslash \Sigma)\} \tag{4.11}$$

is dense in $L_2(\mathbb{R}^d)$.

Suppose that the contrary is true; then there is an element $g \neq 0$ of $L_2(\mathbb{R}^d)$ such that

$$((-\Delta + 1)f, g) = 0 \qquad \text{for all} \quad f \in C_0^\infty(\mathbb{R}^d \backslash \Sigma)$$

This implies that $(f, (-\Delta + 1)g) = 0$ for all $f \in C_0^\infty(\mathbb{R}^d \backslash \Sigma)$, with $(-\Delta + 1)g$ interpreted as a tempered distribution. This equation implies, again in the sense of distribution theory, that $\operatorname{supp}(-\Delta + 1)g \subset \Sigma$. We may assume now that the set Σ is the coordinate plane (x_1, x_2) such that there is a distribution $h(x_1, x_2)$ with

$$[(-\Delta + 1)g](x) = h(x_1, x_2)\delta(x_3) \ldots \delta(x_d)$$

where $\delta(x_i)$, $i = 3, \ldots, d$ is the δ function. Then

$$\begin{aligned} g(x) &= \int (-\Delta + 1)^{-1}(x_1, \ldots, x_d; y_1, \ldots, y_d) h(y_1, y_2) \\ &\qquad \times \delta(y_3) \ldots \delta(y_d) dy_1 \ldots dy_d \\ &= \int (-\Delta + 1)^{-1}(x_1 - y_1, x_2 - y_2, x_3, \ldots, x_d) h(y_1, y_2)\, dy_1\, dy_2 \end{aligned} \tag{4.12}$$

By Fourier transformation

$$\| g \|_2^2 = \int |\hat{h}(k_1, k_2)|^2 \left[\frac{1}{(k_1)^2 + (k_2)^2 + k_\perp^2 + 1} \right]^2 dk_1\, dk_2\, d^{d-2}k_\perp \tag{4.13}$$

where $\hat{h}$ denotes the Fourier transform of h. Thus $\| g \|_2^2 < \infty$, i.e., $g \in L^2(\mathbb{R}^d)$, iff $d - 2 < 4$. This means that, in dimensions larger than or equal to 6, $C_0^\infty(\mathbb{R}^d \backslash \Sigma)$ is a core for $-\Delta$; hence $-\Delta + V = -\Delta$, as self-adjoint operators. This argument is fairly convincing and shows that the quantum mechanical problem of a particle moving in a δ-function potential, concentrated on a typical Brownian path, has critical dimension 6; but see Ref. 18 for a rigorous result from which our claim follows.

We wish to remark that the heretofore known properties of Wegner's nonlinear σ models for the description of the localization transition under renormalization are consistent with the conjecture that, for $d < d_c^{\text{loc}}$,

$$\nu = 1/(d - 2)$$

If true this equation, together with the previously motivated conjecture that $\nu_{\text{classical}} = 1/2$, would imply that

$$d_c^{\text{loc}} = 4$$

and that $\nu = 1$, in three dimensions. For a closely related discussion which motivated this comment see Ref. 19.

To settle these issues the dimension dependence of the infrared properties of Wegner's model should be discussed more carefully.

Finally, we remark that the transport properties in Lloyd's model[20] ($d\lambda$ = Lorentzian) can presumably be studied with the help of a $g(|\phi_1|^2|\phi_2|^2)$ theory, $\phi_i = (\phi_i^1, \ldots, \phi_i^n)$, with complex mass and $g < 0$, in a $n \to 0$ limit. This might also be useful as a tool to make the arguments outlined above more compelling and to determine d_c^{loc}.

NOTE ADDED IN PROOF

We wish to mention an alternative proof of Theorem 2.3.

Theorem. If

$$\left|\int e^{isV} d\lambda(V)\right| \leqslant Ce^{-a|s|}, \qquad \text{with} \quad 2\nu C < a,$$

then $\rho(E)$ is real-analytic in E.

To prove this result, one expands

$$f(s) \equiv \int d\bar{\lambda}(V)\langle x|e^{isH_V}|x\rangle = \int e^{isE}\, d\rho(E)$$

in the off-diagonal part of H_V and exploits the fact that

$$\left|\int d\bar{\lambda}(V) \prod_{j\in M} e^{is_j V(j)}\right| \leqslant C^{|M|} e^{-a\sum_{j\in M}|s_j|}$$

to derive the estimate

$$\begin{aligned} |f(s)| &\leqslant e^{-a|s|} \sum_{n=0}^{\infty} C^{n+1}(2\nu)^n \frac{s^n}{n!} \\ &= C\exp\left[-|s|(a-2\nu C)\right] \end{aligned}$$

from which our claim follows.

This form of Theorem 2.3 was kindly proposed to us and proven by B. Simon. S. A. Molchanov has earlier used closely related methods to prove similar results. In an earlier draft of this paper we have used very similar, but slightly more clumsy, methods with similar conclusions.

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